Sociology 376 Exam 1 Spring 2013 Prof Montgomery

Answer all questions. 200 points possible.

1. [50 pts] Each morning, a child decides whether or not to clean her room (so that the room is either clean or dirty that day). Each evening, the child's parent inspects the room. If the room is clean, the parent doesn't say anything to the child, and there's a 30% chance the child will clean her room the next morning. If the room is dirty that day but was clean the day before, the parent gives the child a gentle reminder, and there's a 50% chance the child will clean her room the next morning. If the room has been dirty for two days in a row (but clean the day before that), the parent gives the child a strong reminder, and there's an 80% chance the child will clean her room the next morning. If the room is dirty for 3 straight days, the parent yells at the child, and the child will clean her room for sure the next day.

a) Can this process be specified as a Markov chain with "clean" and "dirty" as the states of the chain? Explain why or why not.

b) Specify this process as a Markov chain, using the smallest number of states possible. Give the transition diagram and transition matrix.

c) If the child's room was clean on Monday, what's the probability it will be clean on Wednesday?

d) Is the transition matrix primitive? How can you determine primitivity using (only) the transition diagram? Why does primitivity of the transition matrix matter?

e) Over the long run, what proportion of days will the child's room be clean? What proportion of days will the parent yell at the child?

2. [35 pts] A political office can be held by either a Republican (R) or Democrat (D). The incumbent party (which currently holds the office) has a strong advantage. More precisely, if the office is currently held by a Democrat, the next election will be won by a Democrat with probability $1-\varepsilon$ (and won by a Republican with probability ε). If the office is currently held by a Republican, the next election will be won by a Republican with probability $1-4\varepsilon^2$ (and won by a Democrat with probability $4\varepsilon^2$).

a) Over the long run, what proportion of the time is the office held by Republicans?

b) Assuming ε is small, describe the qualitative behavior of the model using the terminology from Peyton Young (in his paper on coordination games).

c) What happens when the incumbency advantage becomes so strong that ε becomes very very small (approaching 0)? How is this different from the outcome when $\varepsilon = 0$ precisely? How is this related to Young's use of "stochastic stability" to predict which Nash equilibrium will be played in coordination games with multiple Nash equilibria?

3. [75 pts] Consider a Markov chain with 6 states: an individual is either (1) taking introductory college courses, (2) taking advanced college courses, (3) employed without a college degree, (4) unemployed without a college degree, (5) employed with a college degree, or (6) unemployed with a college degree. The transition matrix (specified so that element (i,j) is the probability of transitioning from state i to state j) is given by

$$\begin{bmatrix} .4 & .3 & .3 & 0 & 0 & 0 \\ .1 & .5 & 0 & 0 & .4 & 0 \\ 0 & 0 & .7 & .3 & 0 & 0 \\ 0 & 0 & .5 & .5 & 0 & 0 \\ 0 & 0 & 0 & 0 & .8 & .2 \\ 0 & 0 & 0 & 0 & .7 & .3 \end{bmatrix}$$

[NOTE: The matrix is not specified in canonical form.]

a) Find the communication classes, draw the reduced transition diagram, and indicate whether each class is open or closed. [HINT: It isn't required, but it might be helpful to start by drawing the transition diagram. You could find the reachability matrix (by inspection of the transition diagram) as an intermediate step toward the communication classes, or simply find these classes directly (by inspection of the transition diagram).]

b) For a student taking intro courses (state 1), what is the expected time in college (i.e., states 1 and 2)? For a student taking advanced classes (state 2), what is the expected time in college (i.e., states 1 and 2)? To obtain full credit for this part, you must obtain the answer by first computing the fundamental matrix N for this problem (which will also be needed for part c).

[HINT: N is a 2×2 matrix, and A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ implies $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.]

c) For a student taking intro courses, what is the probability of eventually leaving college without a degree (reaching state 3) and the probability of eventually obtaining a college degree (reaching state 5)? For a student taking advanced courses, what is the probability of leaving without a degree (reaching state 3) and the probability of obtaining a college degree (reaching state 5)? To obtain full credit for this part, you must show how to derive the answer from the computation N*R (where N is the fundamental matrix and R is one of the submatrices when the transition matrix is specified in canonical form).

d) Over the long run, for individuals who leave college without a degree, what proportion of their worklife will they be unemployed? For individuals who obtain a degree, what proportion of their worklife will they be unemployed?

e) Suppose the process is modified so that unemployed individuals without a college degree sometimes return to college (taking introductory courses). Redo part (a). How does this modification of the process change your answer to part (d)?

4) [40 points] A population (partitioned into 20-year age classes) has the Leslie matrix below. (Recall the demography convention that L(i,j) reflects population flow *to* age class i *from* age class j.) The eigenvectors and eigenvalues of this matrix are also reported.

```
>> L  % Leslie matrix

L =

         0    1.5000    0.6000         0         0
    0.9500         0         0         0         0
         0    0.9500         0         0         0
         0         0    0.8500         0         0
         0         0         0    0.7000         0

>> [eigvec, eigval] = eig(L)

eigvec =

         0         0   -0.4519   -0.0563   -0.7330
         0         0    0.4710    0.1216   -0.5153
         0         0   -0.4908   -0.2628   -0.3623
         0    0.0000    0.4577    0.5082   -0.2279
    1.0000   -1.0000   -0.3515   -0.8091   -0.1181

eigval =

         0         0         0         0         0
         0         0         0         0         0
         0         0   -0.9116         0         0
         0         0         0   -0.4396         0
         0         0         0         0    1.3512
```

Compute the following:

a) the probability of survival to each age class (from birth)
b) expected number of (20-year) periods remaining for each age class
c) gross reproduction rate (GRR) and net reproduction rate (NRR)
d) equilibrium growth factor
e) equilibrium age distribution (as a probability vector)

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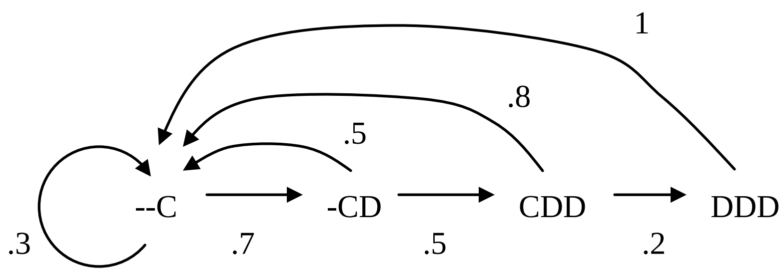
1a) [5 pts] No, you cannot specify the states of the process as “dirty” and “clean.” The transition probabilities given a dirty room depend on the number of days the room has been dirty. Thus, there is too much history dependence for a 2-state specification.

b) [15 pts] Because history dependence extends back 3 periods, and there are 2 possible outcomes (dirty or clean) each day, you could specify a chain with $2^3 = 8$ states. However, it is possible to “lump” some of these states to obtain a chain with only 4 states. One possible specification is

state 1: room was clean the day before (--C)
state 2: room has been dirty for one day (-CD)
state 3: room has been dirty for two days (CDD)
state 4: room has been dirty for three days (DDD)

(Note that the outcomes indicated by the dashes don’t affect the probabilities of transitioning out of the state.) The transition diagram is



and the transition matrix is
$$\begin{bmatrix} .3 & .7 & 0 & 0 \\ .5 & 0 & .5 & 0 \\ .8 & 0 & 0 & .2 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

c) [5 pts] $P^2(1,1) = [.3\ .7\ 0\ 0] * [.3\ .5\ .8\ .1]' = .3 * .3 + .7 * .5 = .44$

d) [10 pts] Yes, the matrix is primitive. Give that the matrix is irreducible (every state can reach every other state), the matrix is primitive if you can find two cycles whose lengths are relatively prime. Here, the loop at node 1 suffices (because a loop is a cycle of length 1 and irreducibility guarantees other longer cycles). Primitivity implies that the process has a unique limiting distribution x given by the solution to x = xP.

e) [15 pts] The condition x = xP can be written as the equations x(1) = .3x(1) + .5x(2) + .8x(3) + x(4) and x(2) = .7x(1) and x(3) = .5x(2) and x(4) = .2x(3). Note that x is a probability vector so that x(1)+x(2)+x(3)+x(4) = 1. Substitution into this equation yields x(1)+.7x(1)+.35x(1)+.07x(1) = 1. Hence x(1) = 1 / 2.12 = .4716 (= proportion of days with a clean room) and x(4) = .07x(1) = .0330 (= proportion of days the parent yells).

2a) [10 pts] We can write the limiting distribution as $x = [p\ 1\text{-}p]$ where p is the proportion of time the Republicans hold office. Note that $p = p\,(1-4\varepsilon^2) + (1-p)\varepsilon$. Assuming $\varepsilon > 0$, we thus obtain $p = 1 / (1+4\varepsilon)$.

b) [10 pts] Adopting Young's terminology, the present model displays "local conformity" (the office is typically held by one party for a long time), "global diversity" (one office might be held by a Democrat for a long time while another office has been held by a Republican for a long time), and "punctuated equilibria" (occasionally, there is an abrupt jump from one party to the other).

c) [15 pts] The result from part (a) indicates that $p \to 1$ as $\varepsilon \to 0$. Thus, if ε is positive but extremely small, Republicans almost always (and Democrats almost never) hold office over the long run. In contrast, if $\varepsilon = 0$, whichever party initially held office will retain the office forever. [Note that the computation in part (a) entailed division by ε, so the result doesn't hold if $\varepsilon = 0$.]

Standard game theory (which assumes that players always choose best responses) offers no way to assess whether one Nash equilibrium of a coordination game is more likely to occur than another Nash equilibrium of that game. By introducing rare "mistakes" (occurring with probability ε) into player's choices, Young shows that some equilibria may be stochastically state (occurring with positive probability as $\varepsilon \to 0$) while other equilibria may not be stochastically stable (occurring with probability $p \to 0$ as $\varepsilon \to 0$).

One interpretation of the present problem is that the incumbent party is kicked out of office only if it makes a rare "mistake." Republicans are twice as likely to make one mistake (probability 2ε versus ε), but more importantly it takes two mistakes to kick out the Republicans but only one mistake to kick out the Democrats. When the probability of mistakes is extremely small, the probability of $D \to R$ is much larger than $R \to D$, and hence the Democrats almost never hold office.

3a) [18 pts] Two states are in the same communication class if they can reach and be reached by each other. For this problem, the communication classes are {1,2}, {3,4}, and {5,6}. The reduced transition diagram is

$$\{3,4\} \longleftarrow \{1,2\} \longrightarrow \{5,6\}$$

Thus, class {1,2} is open and classes {3,4} and {5,6} are closed.

b) [20 pts] Essentially, the closed classes act as absorbing states. Thus, the Q matrix (reflecting transitions from states in the open class to states in the open class) is given by the first two rows and columns of the full transition matrix, and we obtain

$$Q = \begin{bmatrix} .4 & .3 \\ .1 & .5 \end{bmatrix}, \quad I\text{-}Q = \begin{bmatrix} .6 & -.3 \\ -.1 & .5 \end{bmatrix}, \text{ and } N = (I\text{-}Q)^{-1} = \frac{1}{.27}\begin{bmatrix} .5 & .3 \\ .1 & .6 \end{bmatrix} = \begin{bmatrix} 1.85 & 1.11 \\ .37 & 2.22 \end{bmatrix}$$

Thus, a student taking intro courses can expect to spend 2.96 periods in college (sum of first row of N), and a student taking advanced courses can expect to spend 2.59 periods in college (sum of second row of N).

c) [15 pts] For this problem, R is the submatrix reflecting transitions from states in the open class (1,2) to the states in the closed class (3, 4, 5, 6).

$$NR = \begin{bmatrix} 1.85 & 1.11 \\ .37 & 2.22 \end{bmatrix}\begin{bmatrix} .3 & 0 & 0 & 0 \\ 0 & 0 & .4 & 0 \end{bmatrix} = \begin{bmatrix} .555 & 0 & .444 & 0 \\ .111 & 0 & .888 & 0 \end{bmatrix}$$

Thus, a student taking intro courses (state 1) will eventually leave college without a degree (enter state 3) with probability .555, and will eventually leave college with a degree (enter state 5) with probability .444. In contrast, a student taking advanced courses (state 2) will eventually leave without a degree (enter state 3) with probability .111 and will leave with a degree (enter state 5) with probability .888.

d) [10 pts] We can write the limiting distribution as x = [1-p p] where p is the probability of unemployment. For those without degrees, p is determined by the equation p = .3(1-p) + .5p and hence p = 3/8 = .375. For those with degrees, p is determined by the equation p = .2(1-p) +.3p and hence p = 2/9 = .222.

e) [12 pts] The classes are now {1, 2, 3, 4} and {5, 6}. The reduced transition diagram is now {1, 2, 3, 4} $\rightarrow$ {5, 6}. Thus, {1, 2, 3, 4} is open and {5, 6} is closed. In the long run, all individuals have a college degree and spend essentially all of their (infinite) lifetime bouncing between states 5 and 6. Thus, they are unemployed 22% of the time.

[Comment on question 3: If you had access to Matlab, you could have found answers to parts (c) and (d) and (e) from the computations below. But the point was to find the answers efficiently without Matlab.]

```
>> P  % for parts a through d

P =
    0.4000    0.3000    0.3000         0         0         0
    0.1000    0.5000         0         0    0.4000         0
         0         0    0.7000    0.3000         0         0
         0         0    0.5000    0.5000         0         0
         0         0         0         0    0.8000    0.2000
         0         0         0         0    0.7000    0.3000

>> P^1000

ans =
    0.0000    0.0000    0.3472    0.2083    0.3457    0.0988
    0.0000    0.0000    0.0694    0.0417    0.6914    0.1975
         0         0    0.6250    0.3750         0         0
         0         0    0.6250    0.3750         0         0
         0         0         0         0    0.7778    0.2222
         0         0         0         0    0.7778    0.2222

>> P  % for part e, with 10% chance of moving from state 1 from state 4

P =
    0.4000    0.3000    0.3000         0         0         0
    0.1000    0.5000         0         0    0.4000         0
         0         0    0.7000    0.3000         0         0
    0.1000         0    0.5000    0.4000         0         0
         0         0         0         0    0.8000    0.2000
         0         0         0         0    0.7000    0.3000

>> P^1000

ans =
    0.0000    0.0000    0.0000    0.0000    0.7778    0.2222
    0.0000    0.0000    0.0000    0.0000    0.7778    0.2222
    0.0000    0.0000    0.0000    0.0000    0.7778    0.2222
    0.0000    0.0000    0.0000    0.0000    0.7778    0.2222
         0         0         0         0    0.7778    0.2222
         0         0         0         0    0.7778    0.2222
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4a) [10 pts]

$$\begin{bmatrix} 1 \\ .95 \\ .95*.95 \\ .95*.95*.85 \\ .95*.95*.85*.7 \end{bmatrix} = \begin{bmatrix} 1 \\ .95 \\ .9025 \\ .7671 \\ .5369 \end{bmatrix}$$

b) [10 pts]

$$\begin{bmatrix} 1+.95+.95*.95+.95*.95*.85+.95*.95*.85*.7 \\ 1+.95+.95*.85+.95*.85*.7 \\ 1+.85+.85*.7 \\ 1+.7 \\ 1 \end{bmatrix} = \begin{bmatrix} 4.16 \\ 3.32 \\ 2.44 \\ 1.70 \\ 1 \end{bmatrix}$$

c) [10 pts]

GRR = 1.5 + .6 = 2.1
NRR = 1.5 * .95 + .6 * .95 * .95 = 1.9665

d) [5 pts] The equilibrium growth factor is 1.3512, equal to the dominant eigenvalue of the Leslie matrix.

e) [5 pts] The equilibrium age structure is the eigenvector associated with the dominant eigenvalue, normalized to become a probability vector.

$$\begin{bmatrix} -.7330 \\ -.5153 \\ -.3623 \\ -.2279 \\ -.1181 \end{bmatrix} / (-1.9566) = \begin{bmatrix} .3746 \\ .2633 \\ .1851 \\ .1164 \\ .0603 \end{bmatrix}$$

Sociology 376 Exam 2 Spring 2013 Prof Montgomery

Answer all questions. 210 points possible.

1) [75 points] Consider a large population in which each individual decides to either attend or not attend a public event. Each individual is characterized by her threshold level – the attendance rate above which she will attend (and below which she won't attend). Assume that individuals have adaptive expectations, so that last period's actual attendance becomes this period's expected attendance.

a) Suppose that threshold levels are uniformly distributed between 0.2 and 0.6. That is, the probability distribution function (pdf) for thresholds is given by

$$f(x) = \begin{cases} 2.5 & for\ x \in [0.2, 0.6] \\ 0 & otherwise \end{cases}$$

State the generator function giving x_{t+1} (attendance rate in period t+1) as a function of x_t (attendance rate in period t). Draw the cobweb diagram. Determine the steady state(s) and the stability of each state. [NOTE: Your cobweb diagram doesn't need to be perfectly to scale, but does need to be well labeled, and indicate numerically the "kink points" of the generator function. Your generator function should be well specified for all values of x between 0 and 1. You should give numerical answers for the values of the steady states.]

b) The thresholds described in part (a) can be understood as "lower thresholds" above which individuals become willing to attend. To capture overcrowding effects – the event becomes less enjoyable if too many people attend – we might further suppose that each individual also has an "upper threshold" above which she will no longer attend. (Thus, an individual doesn't attend if x is below her lower threshold, does attend if x is between her lower and upper threshold, and doesn't attend if x is above her upper threshold.) Suppose that the pdf for lower thresholds is given by f(x) in part (a), and that upper thresholds are uniformly distributed between 0.5 and 2.5, so that the pdf for upper threshold is given by

$$g(x) = \begin{cases} 0.5 & for\ x \in [0.5, 2.5] \\ 0 & otherwise \end{cases}$$

State the generator function giving x_{t+1} (attendance rate in period t+1) as a function of x_t (attendance rate in period t). Draw the cobweb diagram. Determine the steady state(s) and the stability of each state. [NOTE: Same note as for part (a).]

c) Suppose that the pdf for lower thresholds is again given by f(x) in part (a). But in place of the pdf given by g(x) in part (b), suppose that all individuals have an upper threshold of 0.8 (i.e., every individual would not attend if expected attendance exceeds 80%). Draw the cobweb diagram (taking care to properly label the diagram and indicate numerical "kink points"). What happens to attendance in the long run? Does your answer depend on the initial condition? Briefly explain.

2) [115 points] One version of the predator-prey model is specified as

$$\Delta P = [a - e P - b Q] P h$$

$$\Delta Q = [- c + d P - f Q] Q h$$

where P is the size of the prey population, Q is the size of the predator population, and the parameters (a, b, c, d, e, f, h) are all positive. Following our classroom notation, h can be interpreted as period length.

a) Solve for the P nullcline(s) and the Q nullcline(s). Plot these nullclines under the assumption that a/e < c/d. Then add arrows to indicate the dynamics in each region of the (positive quadrant of the) phase diagram. [NOTE: You should place P on the horizontal axis and Q on the vertical axis. Your graphs don't need to be perfect, but should be well labeled and qualitatively correct and indicate horizontal intercepts of the nullclines. You can restrict attention to the positive quadrant because the context presumes that both P and Q are non-negative.]

b) Suppose an initial condition with P relatively large and Q relatively small but positive. (More precisely, suppose that $[-(c/f) + (d/f) P] > Q > 0$.) Assuming h is small, what trajectory do these populations follow over time? What is the long-run equilibrium?

c) Now consider this model when a = 5, c = 3, b = d = e = f = 1, and h = 0.5. Plot the nullclines and add arrows to indicate dynamics in each region of the (positive quadrant of the) phase diagram. [HINT: These parameter values imply that a/e > c/d, in contrast to part (a).]

d) Given the parameter values from part (c), what are the steady state(s)? Based on the arrows, can you determine stability of these state(s)? Briefly discuss.

e) Given the parameter values from part (c), find the Jacobian matrix. Then use this matrix to assess the stability of the steady state(s). [HINTS: You should first rewrite the equations so they're in the form $P_{t+1} = F(P_t, Q_t)$ and $Q_{t+1} = G(P_t, Q_t)$. You can then find the elements of the Jacobian matrix using either calculus or non-calculus approaches. It will then help to know that

the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has eigenvalues $\lambda_1 = (1/2)(a + d + \text{sqrt}(a^2 + 4bc - 2ad + d^2))$

$\lambda_2 = (1/2)(a + d - \text{sqrt}(a^2 + 4bc - 2ad + d^2))$

and, for any complex number a+bi (where i = sqrt(-1)), abs(a+bi) = sqrt(a^2+b^2).]

3) [20 points] In question 2 above and throughout the second half of the course, we have often included a parameter h reflecting period length. Why is this necessary? What sorts of behaviors arise in one-dimensional and two-dimensional non-linear models when h is relative large that don't arise when h is very small? Give some examples from the course.

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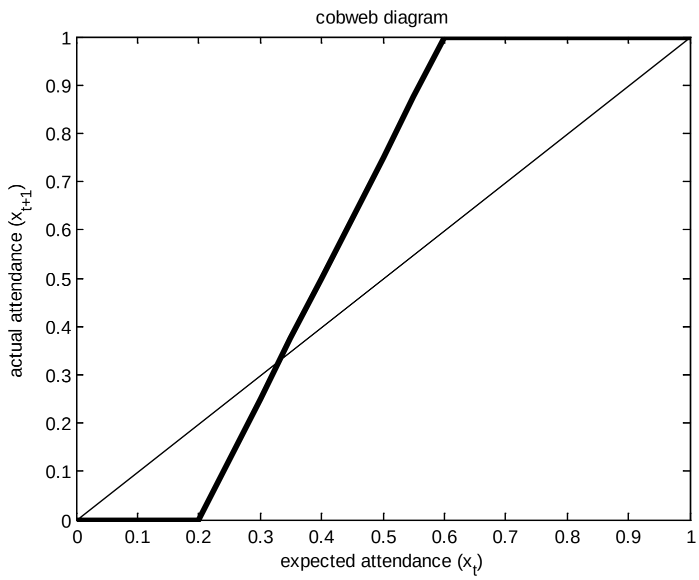
1a) [25 pts] The generator function is the cumulative distribution function (cdf) of thresholds:

$$F(x) = \begin{cases} 0 & for\ x < 0.2 \\ 2.5x - 0.5 & for\ x \in [0.2, 0.6] \\ 1 & for\ x > 0.6 \end{cases}$$

Plotting F(x) against the 45-degree line, we obtain the cobweb diagram:



The diagram reveals stable equilibria at x* = 0 and x* = 1, and an unstable equilibrium at x* = 1/3. (An equilibrium is stable iff the absolute value of the slope of the generator function is less than one. The interior equilibrium is determined by 2.5x* – 0.5 = x* which implies x* = 1/3.)

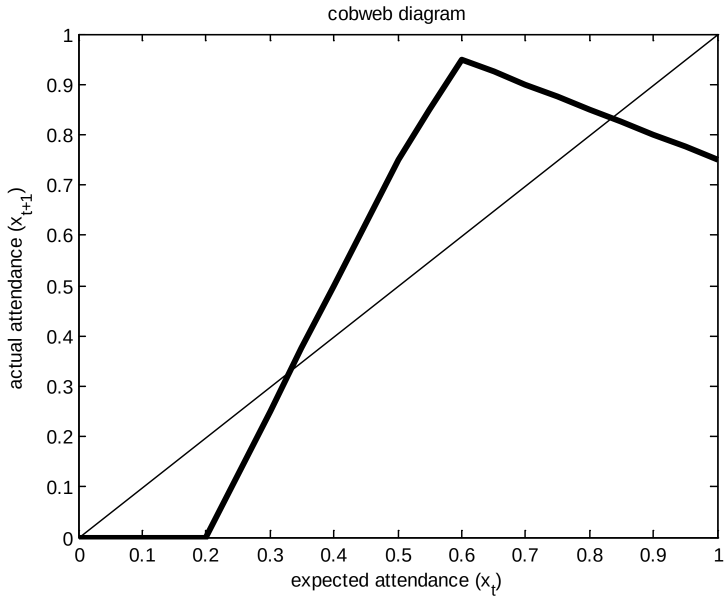
b) [35 pts] The cdf for upper thresholds is given by

$$G(x) = \begin{cases} 0 & for\ x < 0.5 \\ 0.5x - 0.25 & for\ x \geq 0.5 \end{cases}$$

and hence the generator function is given by

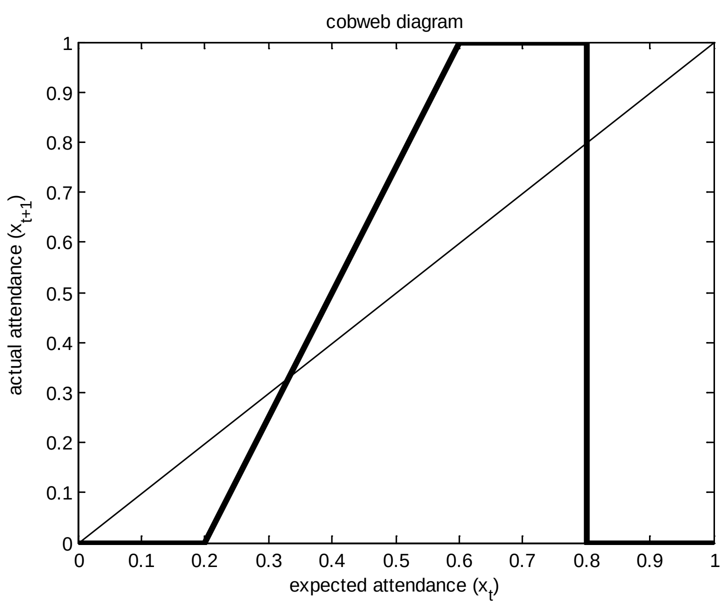
$$F(x) - G(x) = \begin{cases} 0 & for\ x < 0.2 \\ 2.5x - 0.5 & for\ x \in [0.2, 0.5] \\ 2x - 0.25 & for\ x \in [0.5, 0.6] \\ -0.5x + 1.25 & for\ x > 0.6 \end{cases}$$

Plotting the generator function against the 45-degree line, we obtain the cobweb diagram:



The equilibria are x* = 0 (stable), x* = 1/3 (unstable), and x* = 5/6 (stable). (The latter equilibrium is determined by the condition -0.5x* + 1.25 = x*, and is unstable because the slope of the generator function is –0.5, which is less than 1 in absolute value.)

c) [15 pts] From the cobweb diagram, x* = 0 is the only stable equilibrium. Almost every initial condition (the unstable equilibria x* = 1/3 and x* = 0.8 are exceptions) lies on a trajectory that leads ultimately to zero attendance. For $x_0 < 1/3$, attendance will fall monotonically to 0. For $x_0 > 1/3$, attendance will rise until it exceeds 0.8, and then fall to 0 the next period (and remain at 0 thereafter).



2a) [30 pts] The P-nullclines are determined by the condition $\Delta P = 0$ which implies

$$Q = (a/b) - (e/b)\,P \qquad \text{or} \qquad P = 0 \text{ (i.e., the vertical axis is a P-nullcline)}$$

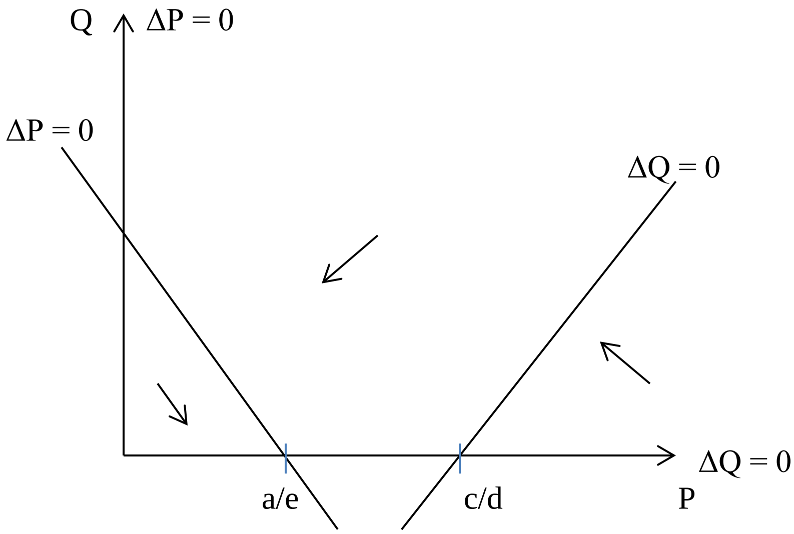
The Q-nullclines are determined by the condition $\Delta Q = 0$ which implies

$$Q = -(c/f) + (d/f)\,P \qquad \text{or} \qquad Q = 0 \text{ (i.e., the horizontal axis is a Q-nullcline)}$$

Dynamics in each region of the phase diagram are given by

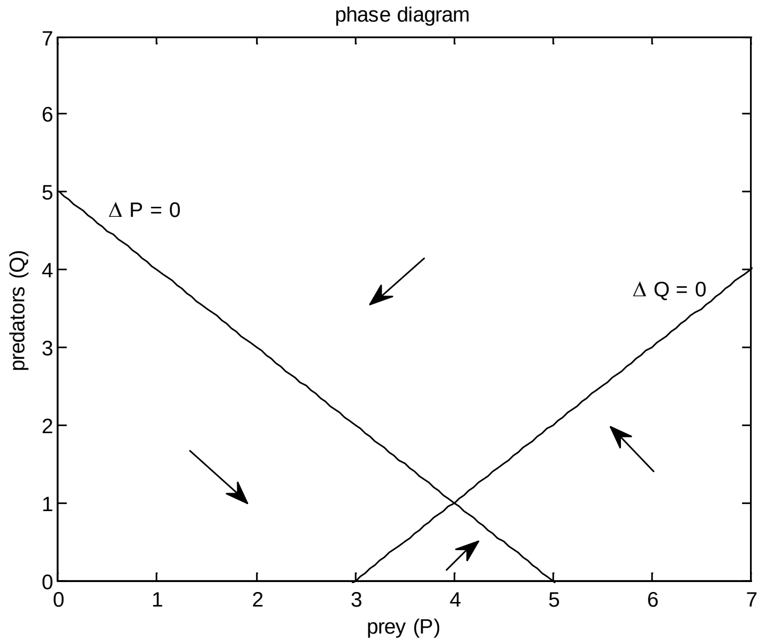
$\Delta P > 0$ implies $Q < (a/b) - (e/b)\,P$
$\Delta Q > 0$ implies $Q < -(c/f) + (d/f)\,P$

Graphically,



b) [10 pts] Given an initial condition with high P and low Q (i.e., a point in the lower right area of the phase diagram), P would fall and Q would initially rise. Once the trajectory crosses the Q-nullcline, both P and Q would fall. The population will eventually converge to the (unique, stable) equilibrium with $P^* = a/e$ and $Q^* = 0$. Intuitively, there aren't enough prey to support a predator population, and the prey population rises to the carrying capacity of the environment.

c) [23 pts] The P-nullcline is now Q = 5 – P, and the Q-nullcline is now Q = –3 + P. The phase diagram is plotted below.



d) [12 pts] Recalling that the vertical axis is a P-nullcline and the horizontal axis is a Q-nullcline, there are three equilibria: (P* = 0, Q* = 0), (P* = 5, Q* = 0) and (P* = 4, Q* = 1). From the phase diagram, it is clear that the first two equilibria are unstable. At (P* = 0, Q* = 0), a small increase in P would cause P to rise further. At (P* = 5, Q* = 0), a small increase in Q would cause Q to rise further. For the interior equilibrium (P* = 4, Q* = 1), it is clear that trajectories will spiral around the equilibrium, but you can't tell from the phase diagram whether this equilibrium is stable (with trajectories spiraling inward) or unstable (with trajectories spiraling outward).

e) [40 pts] You can rewrite the equations (from "delta form") as

$$P_{t+1} = F(P_t, Q_t) = P_t + [5 - P_t - Q_t]\ P_t\,(0.5) = 3.5\,P_t - 0.5\,P_t^2 - 0.5\,P_t\,Q_t$$

$$Q_{t+1} = G(P_t, Q_t) = Q_t + [-3 + P_t - Q_t]\ Q_t\,(0.5) = -0.5\,Q_t + 0.5\,P_t\,Q_t - 0.5\,Q_t^2$$

Using the non-calculus approach to obtain the elements of the Jacobian matrix, suppose the system is initially in equilibrium, and there is a very small shock in period t. Let p_t and q_t denote the deviations from steady state in period t. Given $P_t = P^* + p_t$ and $Q_t = Q^* + q_t$, we obtain

$$P^* + p_{t+1} = 3.5\,(P^* + p_t) - 0.5\,(P^* + p_t)^2 - 0.5\,(P^* + p_t)\,(Q^* + q_t)$$

$$Q^* + q_{t+1} = -0.5\,(Q^* + q_t) + 0.5\,(P^* + p_t)\,(Q^* + q_t) - 0.5\,(Q^* + q_t)^2$$

Simplification (from the definitions of equilibrium and the fact that any second-order terms p_t^2 or q_t^2 or $p_t q_t$ are so small they can be ignored) ultimately yields

$$\begin{bmatrix} p_{t+1} \\ q_{t+1} \end{bmatrix} = \begin{bmatrix} 3.5 - P^* - 0.5\,Q^* & -\,0.5\,P^* \\ 0.5\,Q^* & -0.5 + 0.5\,P^* - Q^* \end{bmatrix} \begin{bmatrix} p_t \\ q_t \end{bmatrix}$$

where the (2×2) matrix is the Jacobian matrix. Using calculus, you could have found the Jacobian matrix directly as

$$\text{J} = \begin{bmatrix} \partial F/\partial P & \partial F/\partial Q \\ \partial G/\partial P & \partial G/\partial Q \end{bmatrix} = \begin{bmatrix} 3.5 - P - 0.5\,Q & -0.5\,P \\ 0.5\,Q & -0.5 + 0.5\,P - Q \end{bmatrix}$$

In order to assess stability, you need to evaluate J at the equilibrium (P*, Q*) under consideration, and then compute the eigenvalues. The equilibrium is stable if the absolute value of the leading eigenvalue is less than 1.

For this problem, there are three equilibria to consider:

(P* = 4, Q* = 1) implies $\text{J} = \begin{bmatrix} -1 & -2 \\ 0.5 & 0.5 \end{bmatrix}$ implies $\lambda = -\,0.25 \pm 0.66\,i$

(P* = 5, Q* = 0) implies $\text{J} = \begin{bmatrix} -1.5 & -2.5 \\ 0 & 2 \end{bmatrix}$ implies $\lambda = 0.25 \pm 1.75$

(P* = 0, Q* = 0) implies $\text{J} = \begin{bmatrix} 3.5 & 0 \\ 0 & -0.5 \end{bmatrix}$ implies $\lambda = 1.5 \pm 2$

Because abs(– 0.25 ± 0.66 i) = 0.706 which is less than 1, the first equilibrium is stable. Because abs(2) > 1, the second equilibrium is unstable. Because abs(3.5) > 1, the third equilibrium is unstable.

3) [20 pts] To minimize use of calculus, our class has focused on "discrete-time" models in which state variables are updated at discrete time steps (t, t+1, t+2, …). Arguably, some social processes are better specified using "continuous-time" models in which state variables change continuously through time. By introducing the period-length parameter h, and then setting h relatively small, we can approximate continuous-time dynamics using a discrete-time model.

Phase diagrams (which include arrows to indicate direction of flow) often provide a good guide to assess stability of steady states in continuous-time models, but can be misleading for discrete-time models. In one-dimensional continuous-time models, trajectories will flow smoothly into stable steady states, and cycles (or chaos) are not possible. In contrast, for one-dimensional discrete-time models, trajectories can "jump over" steady states, and the model can display cycles or chaotic dynamics. (Consider the logistic population model studied in class.) In two-dimensional models, a steady state which would be stable in continuous time may be unstable in discrete time. (Consider the Homans-Simon or predator-prey models studied in class.)